Managerial Flexibility, Agency Costs and Optimal Capital Structure\footnote{Preliminary version; comments are most welcome and will be greatly appreciated; especially references to work that has inadvertently not been cited.}

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Abstract

This paper investigates the effect of managerial flexibility on the choice of capital structure for a firm and the corresponding valuation of its long-term debt. We consider a general model in continuous time where the manager of a firm with long term debt in place may dynamically change firm strategies at random times to maximize his discounted expected utility. The manager is compensated with the commonly observed structure of a base salary and executive options on firm assets. He may bear personal costs due to liquidation of the firm and the firm enjoys a tax shield on its interest payments to creditors. Under general assumptions, we show the existence of and derive explicit analytical characterizations for the optimal policies for the manager. We use these analytical results to endogenously derive the optimal option compensation structure for the manager. We then derive the optimal policies for the firm that can hypothetically completely contract for managerial behavior ex ante, i.e. before debt is in place. We investigate the implications of these results for the optimal capital structure for the firm in the presence of managerial flexibility and the valuation of its long-term debt. We also obtain precise quantitative characterizations of the agency costs of debt due to managerial flexibility in a very general context and show that they are very significant when compared with the tax advantages of debt thereby implying that managerial flexibility is a very important determinant of the choice of optimal capital structure for a firm. We carry out several numerical simulations with different choices of underlying parameter values to calculate the optimal leverage, agency costs, corporate debt values and bond yield spreads and study the comparative statics of these quantities with respect to the parameters characterizing the strategies available to the manager. The optimal leverage levels predicted by our model correspond well with average leverage levels observed in the marketplace.

Key Words: Managerial Flexibility, Agency Theory, Capital Structure, Default Risk

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Introduction

The importance of agency theory in understanding the investment and financing decisions of firms has been understood and accepted at least since the seminal paper of Jensen and Meckling (J-M) [1976]. The paper critically analyzed the Modigliani-Miller [1958] assumption of independence of the investment and financing decisions for a firm. The paper argued persuasively that equity holders of a leveraged firm, for example, could make use of asset substitution to extract value from bondholders after debt is in place. This phenomenon creates agency costs that must be controlled for thereby forging an inextricable link between the capital structure of the firm and its investment decisions. However, increased risk-taking may limit the ability of the firm to shield itself from taxes through leverage so these conflicting forces would seem to indicate that there is, in fact, an optimal amount of debt that a firm should issue, and hence, an optimal capital structure for a firm.

The research of Jensen and Meckling spawned a large amount of theoretical and empirical research investigating how the choice of capital structure is influenced by these considerations. However, until recently, there were no attempts to completely integrate and synthesize the different approaches within one realistic unified framework. This is an important prerequisite for being able to accurately evaluate the relative importance of the different economic factors that affect the value of a firm.

This problem has been addressed by several recent papers\(^3\). Using models differing in some important underlying assumptions, Leland [1998] and Ericsson [2001] succeeded in precisely quantifying how the capital structure of a firm is influenced by ex-post flexibility of shareholders in choosing risk and how the presence of debt distorts a firm’s ex-post choice of risk. In particular, the papers succeeded in valuing debt and leveraged equity in the presence of firm risk-taking thereby obtaining a precise characterization of the agency costs of debt associated with asset substitution by

equity holders. However, in spite of the considerable progress that has been made, there are several questions that remain unanswered.

Leland [1998] and Ericsson [2001] both implicitly assume that the manager of the firm always behaves in the interests of shareholders. With the recent spate of corporate debacles in the U.S. industry, it has become painfully apparent that in many cases, managers do not seem to behave in shareholders' interests. Graham and Harvey [2001] report that the single most important factor affecting debt decisions of firms is management's desire for financial flexibility. It has become quite clear that in many cases, it is this flexibility enjoyed by managers in dictating firm policy that has led to the erosion of the value of several firms.

This paper proposes a dynamic model in continuous time that addresses several different, but closely related issues, within a realistic unified framework. Our primary goal is to precisely quantify the importance of managerial flexibility in a firm's choice of leverage and the corresponding valuation of its long-term debt. The manager has ex-post (i.e. after debt is in place) dynamic flexibility in choosing firm strategies, but may also bear significant personal costs if the firm is liquidated. We assume that the manager is compensated with the commonly observed structure of a base salary and executive options on firm assets. Therefore, we also simultaneously address the problems of determining the cost of executive option compensation to the firm and measuring its success in aligning managerial interests with those of shareholders.

We achieve these goals by first investigating the optimal asset substitution problem for the manager in continuous time. We begin by assuming that the manager’s compensation structure is exogenously specified. He is periodically compensated with a fixed base salary and options on firm assets. The manager may dynamically choose between strategies with different expected returns and volatilities over an infinite time horizon so as to maximize his expected utility of compensation\(^4\). We explicitly derive the optimal dynamic policies for the manager under very general assumptions on the nature of the available strategies. The explicit characterization of managerial optimal policies allows us to easily derive analytical expressions for the values of the firm, its debt and its equity under additional assumptions on the form of debt service. We then *endogenously* derive the optimal option compensation structure for the manager as that which maximizes shareholders’ value of equity. We use these results to derive the optimal capital structure of the firm in the presence of

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\(^4\) This is in sharp contrast with the model of Ericsson [2001] where risk shifting can only occur once and is irreversible, but is in conformity with the model of Leland [1998].
managerial flexibility and the corresponding valuation of the firm’s debt. We therefore characterize
the partial equilibrium between the firm and the manager, i.e. the manager is optimally compensated
within the class of option compensation structures and the firm chooses its capital structure
optimally, rationally anticipating ex-post managerial flexibility.

We then study the optimal asset substitution problem for the firm that can hypothetically
completely contract for managerial behavior ex ante, i.e. before debt is in place. We explicitly derive
the optimal policies for the firm in this situation and elucidate the nature of the conflict between the
manager’s interests and those of the firm. These investigations and the explicit analytical results
allow us to obtain precise quantitative characterizations of the agency costs associated with
managerial flexibility by employing the measure proposed by Leland [1998], i.e. we compare the
hypothetical ex ante situation where the firm can completely contract for the policies to be followed
by the manager and the real ex-post situation where managerial policy can only be controlled
through his compensation contract. Through several numerical simulations, we demonstrate that,
when the manager is compensated with executive options, the agency costs of managerial flexibility
are very significant when compared with the tax advantages of debt thereby significantly affecting
the leverage of the firm.

These results should be contrasted with the results of Leland’s numerical simulations that
indicate that the agency costs of debt associated with shareholder asset substitution are very
insignificant when compared with the tax advantages of debt and that, therefore, shareholder asset
substitution is not a significant determinant of the capital structure of the firm. We thereby
conclusively demonstrate that managerial flexibility is a far more significant determinant of capital
structure of the firm than asset substitution driven by shareholders’ interests. This theoretical result
is consistent with the empirical findings of Graham and Harvey [2001].

The existence of analytical expressions for the manager’s optimal policies also allows us to
easily investigate the dependence of managerial compensation, capital structure, agency costs, and
firm value on the menu of strategies available to the manager. Moreover, our framework is broad
and general enough to also obtain an accurate quantitative characterization of the costs of
asymmetric information between the manager and the firm’s investors where the investors only
know the risks (or volatilities) of the strategies available to the firm, but not their expected returns.

In summary, this paper proposes and investigates a unified and general framework that
provides answers to the following questions:
1. Given that the manager of a leveraged firm has the flexibility to choose between different strategies with different risks and expected returns, what is his optimal dynamic ex-post policy (i.e. after debt is in place) for a given exogenously specified incentive compensation structure consisting of a base salary and options on firm assets?

2. What are the costs of option compensation to the firm and what is the optimal option compensation structure for the manager?

3. What are the implications of these considerations for the valuation of the firm and therefore, its choice of capital structure?

4. How significant are the agency costs of managerial flexibility?

5. How are the firm’s optimal leverage, agency costs and the yield spreads on corporate bonds affected by the menu of strategies available to the manager?

6. What are the costs of asymmetric information between the manager and the firm’s investors?

Our detailed analysis of the optimal asset substitution problem for the firm that can completely contract for managerial behavior ex ante reveals that the firm will not always prefer higher risk close to liquidation. We provide a necessary and sufficient condition for the firm to increase risk close to liquidation. In contrast, if it is optimal for the manager to switch strategies, he will always prefer to decrease risk as the firm value declines towards the liquidation level and we provide necessary and sufficient conditions for this to occur. These results illustrate the implications of the conflict between the manager’s and the firm’s interests.

We show that for reasonable choices of parameter values, the agency costs of debt due to managerial flexibility may be as high as 60% of the tax advantages of debt and the optimal leverage of the firm may be as much as 20% lower than it would be in the absence of managerial flexibility. Managerial flexibility is therefore a very significant determinant of the choice of capital structure for a firm.

Apart from the specific problem that is investigated in this paper, the results of the paper are, we believe, of independent interest since they have implications in a much more general context. Since we propose and solve the problem of optimal asset substitution in continuous time for a manager with an option-like compensation structure, our results can be directly applied to the investigation of the problem of deriving the optimal policies for the portfolio manager of a mutual fund (or hedge fund) in a general incomplete market with a convex compensation structure.
The plan for the paper is as follows. In Section 1, we present the model. In Section 2, we derive the optimal policies for the manager. In Section 3, we derive the optimal policies for the firm that can hypothetically completely contract for managerial behavior. In Sections 4 and 5, we investigate the implications of the results of Sections 2 and 3 for the optimal capital structure for the firm in the presence of managerial flexibility, the associated agency costs and present numerical results that study their dependence on the nature of the available asset substitution strategies. Section 6 concludes the paper. All detailed proofs appear in the Appendix.

1. The Model

Throughout the paper, we consider a filtered probability space \((\Omega, F, P, F_t)\) with the filtration \(F_t\) (completed and augmented) generated by two independent Brownian motions \(B_1, B_2\). A firm has a certain amount of long-term debt in place that is completely amortized, i.e. the firm is liable for an interest (coupon) payment of \(J\) per unit time over an infinite time horizon. Initially, we assume that the interest rate \(J\) is exogenously specified. Later, we will endogenously determine the optimal interest rate and therefore the optimal leverage of the firm in the presence of managerial flexibility.

The manager of the firm is assumed to be risk-neutral and need not behave in the interests of the firm’s investors, i.e. shareholders and bondholders. The manager is periodically compensated with a fixed base salary and options on firm assets with fixed maturities and strike prices. Throughout the paper, we assume that his compensation is negligible compared with the cash flows from the firm’s operations. His discount factor or opportunity cost of capital is \(\beta\) so that the discount factor for cash flows at time \(t\) is \(\exp(-\beta t)\).

In this section, we assume that the manager’s compensation structure is exogenously specified. Later, we endogenously derive the optimal incentive compensation structure for the manager as that which maximizes the value of equity. We assume that the manager’s compensation contract is not renegotiated and that he is retained as long as the firm is in operation. The manager’s goal is to choose firm strategies so as to maximize the (discounted) expected utility of cash flows.

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5 We may easily extend the results of the paper to the situation where the manager’s compensation is significant when compared with firm cash flows, but this introduces additional technicalities without substantially altering the import of our results.
comprising his compensation. In contrast with Leland [1998], we assume that the firm does not retire its debt or restructure it at intermediate times\(^6\).

\(P(.)\) is the process for a state variable that determines the cash flows from the firm’s operations. We assume (as is usually done in the extant literature) that the cash flows associated with the firm’s operations are spanned by marketed securities. The cash flows (per unit time) arising from the firm’s operations (before coupon payments to creditors) are proportional to the value of the state variable \(P(.)\) and equal to \(\lambda P(t)\) at time \(t\). \(P(.)\) is the price process of a traded asset that has a cash payout ratio of \(\delta\) per unit time.

We assume that there is an exogenous level \(p_b \leq \frac{\delta}{\lambda}\) of the state variable \(P(.)\) at which the firm is liquidated. The liquidation level \(p_b\) may be a function of the coupon rate \(\delta\). The costs of liquidation are proportional to the value of the firm’s assets. As long as the firm’s cash flows exceed the required interest payments, they are used to service debt. If they are lower than the required interest payments, the shareholders of the firm inject capital to service debt as long as the value of equity is positive. If there is an (endogenous) level \(p_e > p_b\) at which the value of equity falls to zero and the value of the state variable \(P(.)\) lies between \(p_e\) and \(p_b\), the bondholders make concessions to shareholders by accepting all cash flows from the firm’s operations in lieu of interest payments\(^7\).

Our assumptions of exogenous liquidation and incomplete debt service differ from those of Leland [1998] who assumes that debt is serviced entirely during financial distress and transfer of control occurs as soon as the value of equity falls to zero. They are inspired by the results on strategic debt service obtained by Mella-Barral and Perraudin [1997]\(^8\). In their model, they show that if shareholders can make “take it or leave it” offers to bondholders (and direct bankruptcy costs are negligible) OR, if bondholders can make “take it or leave it” offers to shareholders (even in the presence of significant direct bankruptcy costs), it is optimal for bondholders and shareholders in both situations to agree to the following form of debt service: shareholders inject capital to service debt entirely as long as the value of equity is positive and bondholders inject capital into the firm in the region between the level where the value of equity first falls to zero and the level where the firm

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\(^6\) It is possible to use Leland’s procedure to allow dynamic debt restructuring, but we do not consider it in this paper so that we can focus on our central issue of managerial flexibility.

\(^7\) Since \(p_b \leq \frac{\delta}{\lambda}\), the cash flows are clearly less than the contracted interest payments.

\(^8\) See Anderson and Sundaresan [1998] for another model of strategic debt service.
is liquidated. Therefore, it is sub-optimal for transfer of control to actually take place and the value of equity remains the same with or without renegotiation of debt payments. This result translates to the form of debt service we have assumed within the context of our model. It is important to emphasize that the form of debt service assumed represents a redistribution of wealth between bondholders and shareholders that does not affect overall firm-related cash flows.

There are of course several possible choices for the exogenous liquidation trigger level \( p_b \). It could be the level at which net earnings after interest fall below zero (Kim, Ramaswamy and Sundaresan 1993) or it could be the level at which the value of the unlevered assets of the firm falls below the outstanding debt principal. As in Mella-Berral and Perraudin [1997], it could also be the level at which shareholders of an all-equity financed firm would optimally choose to liquidate the firm. In order to keep our treatment as general as possible and to avoid unnecessary analytical complications, we assume that liquidation occurs at some level \( p_b \) that may be a function of the coupon rate \( \delta \). This general framework has the additional advantage of allowing us to investigate the comparative statics of the manager’s optimal compensation, the agency costs of debt and the optimal capital structure of the firm with the exogenous liquidation level.

We assume that the manager bears significant personal costs due to liquidation of the firm consistent with the findings of several empirical papers\(^9\). The results of the paper hold if the manager bears any strictly nonzero costs due to liquidation of the firm. For notational and expositional convenience, we however assume throughout that the manager is replaced with no severance compensation.

**Available Strategies for the Manager**

At any instant of time, the manager of the firm can switch between two strategies *without cost*. The state variable \( P(.) \) evolves in the *real world* as follows under the two strategies\(^10\):

\[
\begin{align*}
(1a) \quad dP(t) &= P(t)[(\mu_1 - \delta)dt + \sigma_1 dB_1(t) + \sigma_1 dB_2(t)] \quad \text{Strategy 1} \\
&= P(t)[(\mu_2 - \delta)dt + \sigma_2 dB_1(t) + \sigma_2 dB_2(t)] \quad \text{Strategy 2}
\end{align*}
\]

Therefore, we may rewrite equations (1a) as follows:

\(^9\) See e.g. Mehran [1992].

\(^10\) The state variable processes under the two strategies need not be perfectly correlated with each other.
\( dP(t) = P(t)[\mu_1 dt + \sigma_1 dB_1^*(t)] \) Strategy 1
\( dP(t) = P(t)[\mu_2 dt + \sigma_2 dB_2^*(t)] \) Strategy 2

where \( B_1^*, B_2^* \) are (not necessarily perfectly correlated) \( F_t \)-Brownian motions with
\[
\sigma_1 = \sqrt{\sigma_{11}^2 + \sigma_{12}^2}, \mu_1 = \mu_1 - \delta, \sigma_2 = \sqrt{\sigma_{21}^2 + \sigma_{22}^2}, \mu_2 = \mu_2 - \delta.
\]
Thus, if the manager has initially chosen strategy 1 and switches to strategy 2 at time \( t^* \), then the evolution of the state variable \( P(.) \) for times \( t > t^* \) is described by the drift and volatility parameters \( (\mu_2, \sigma_2) \) until it switches back to strategy 1 in which case the evolution is governed by the drift and volatility parameters \( (\mu_1, \sigma_1) \).

Therefore, the state variable process \( P(.) \) is always continuous. We assume that \( \mu_1, \mu_2, \sigma_1, \sigma_2 \) are constants, \( \beta > \mu_1 > \mu_2 > 0 \) and \( \sigma_1 > \sigma_2 \), but don’t make any further assumptions on their values.

Thus, strategy 1 has a higher drift (or higher expected return) and higher volatility than strategy 2 that is consistent with the usual tradeoff between expected return and variance\(^{12}\). Strategy 1 could be interpreted as a high growth strategy and strategy 2 could be interpreted as a conservative strategy for the firm.

Therefore, the manager’s policies \( \Gamma \) may be described as follows:
\[
\Gamma \equiv \{\tau_1, \tau_2, \tau_3, \ldots\}
\]
where \( \tau_i \) are increasing \( F_t \)-stopping times (reflecting the fact that the manager’s decisions cannot anticipate the future) representing the instants where the manager switches strategies. The goal of the manager is to choose his policy to maximize his expected discounted compensation that is given by
\[
U_t(p) = E\left[\int_0^{\tau_0} \exp(-\beta t) f dt + \int_0^{\tau_5} \exp(-\beta t)[\lambda P_t(t) - q]^+ dt\right],
\]
where \( f \) is the fixed salary per unit time and the second term is the discounted expected variable compensation comprising of options on firm cash flows. We make the simplifying assumption that

\(^{11} It is easy to see that if \( \beta < \mu_1 \), the value function for the manager’s optimization problem (3) is infinite.

\(^{12} When \( B_1^*, B_2^* \) are perfectly correlated, then the market prices of risk of the two strategies must be equal to prevent arbitrage in which case \( \mu_1 > \mu_2 \Rightarrow \sigma_1 > \sigma_2 \).
the manager is periodically compensated with a fixed number of executive options with the same
strike and maturity and that these are exercised at maturity.\textsuperscript{13}

\[ g \] is the number of options the manager obtains (and exercises) per unit time. \( P_t(\cdot) \) is the
state variable process when the manager follows strategy \( \Gamma \) described in (2) and \( \tau_b \) is the time
(stopping time) at which liquidation occurs.\textsuperscript{14} As mentioned earlier, we assume that the manager’s
compensation structure described by the quantities \( f, g, q \) is \textit{exogenously specified}. Our explicit
derivation of the optimal policies of the manager given this exogenous compensation structure in the
next section allows us to later derive the optimal compensation structure for the manager, i.e. we
\textit{endogenously} obtain the parameters \( f, g, q \) that maximize shareholders’ value of equity.

If \( u(\cdot) \) is the value function of the dynamic optimization problem (3), then we can use
traditional dynamic programming arguments (see e.g. Oksendal 1998) to write down the following
formal Hamilton-Jacobi-Bellman equation for \( u \) :

\begin{equation}
- \beta u + \sup_{\Gamma=1,2} \left[ \mu_p u_{p} + \frac{1}{2} \sigma^2_p p^2 u_{pp} \right] + f + g(\lambda p - q)^+ = 0, \; p \in (p_b, \infty)
\end{equation}

\[ u(p_b) = 0 \]

where \( u(p_b) = 0 \) above represents the boundary condition for the manager’s value function \( u \) at the
liquidation level \( p_b \). In the dynamic programming framework, the variable \( p \) above represents the
value of the state variable \( P(\cdot) \) so that the term \( f + g(\lambda p - q)^+ \) is the instantaneous rate of
compensation of the manager. Therefore, in regions where it is optimal for the manager to choose
strategy 1, we would expect to have

\begin{equation}
L^1(u) + f + g(\lambda p - q)^+ = 0 \quad \text{where} \quad L^1(u) = - \beta u + \mu_p u_{p} + \frac{1}{2} \sigma^2_p p^2 u_{pp},
\end{equation}

and in regions where it is optimal for the manager to choose strategy 2, we would have

\begin{equation}
L^2(u) + f + g(\lambda p - q)^+ = 0 \quad \text{where} \quad L^2(u) = - \beta u + \mu_p u_{p} + \frac{1}{2} \sigma^2_p p^2 u_{pp},
\end{equation}

\textsuperscript{13} Strictly, most executive options are American. This, however, complicates our basic model considerably without
contributing substantially to the underlying intuition.

\textsuperscript{14} The compensation structure described by (3) also includes as special cases the linear compensation structure (option
strike price = 0) or the traditional compensation structure of a base salary and a bonus proportional to firm profits net of
interest payments (option strike price = interest rate). .
We shall now state without proof the following standard verification result.

**Proposition 1**

If \( u : [p_b, \infty) \to R_+ \) is a continuous function that is twice differentiable on \((p_b, \infty)\) satisfying the HJB equation (4), then \( u \) is the value function of the manager’s optimization problem (3).

**Proof.** See e.g. Oksendal [1998].

This completes the formulation of the model and the mathematical preliminaries.

**2. Optimal Policies for the Manager**

In this section, we shall show the existence of and explicitly derive the optimal policies for the manager for all possible pairs of strategies 1 and 2 characterized by the drift-volatility parameters \((\mu_1, \sigma_1)\) and \((\mu_2, \sigma_2)\) satisfying \( \beta > \mu_1 > \mu_2 > 0 \) and \( \sigma_1 > \sigma_2 \). We show that

- It is EITHER optimal for the manager to choose strategy 1, i.e. the high risk strategy always
  or optimal for the manager to choose strategy 2 close to the liquidation trigger and switch to strategy 1 when the value of the firm’s assets moves away from the liquidation trigger.
- We provide analytical necessary and sufficient conditions for each of the two cases above.

We shall begin by introducing two quadratic equations that are the characteristic equations of the generators \( L^1, L^2 \) defined in (5) and (6).

\[
\begin{align*}
\frac{1}{2} \sigma_1^2 x^2 + (\mu_1 - \frac{1}{2} \sigma_1^2) x - \beta &= 0 \\
\frac{1}{2} \sigma_2^2 x^2 + (\mu_2 - \frac{1}{2} \sigma_2^2) x - \beta &= 0
\end{align*}
\]

Each of the equations above has two real roots, one of which is strictly positive and the other strictly negative. Let us denote the positive and negative roots of the equations above by \( \eta_1^+, \eta_1^- \) and \( \eta_2^+, \eta_2^- \) respectively. Throughout the paper, we shall assume that \( \eta_1^+ \neq \eta_2^+, \eta_1^- \neq \eta_2^- \), i.e. the available
strategies are such that the roots of equations (10) are all distinct\textsuperscript{15}. We can now prove the following lemma that collects properties of the roots of equations (7) that we will use frequently.

**Lemma 1**

\(1\) \(1 < \eta_i^+ < \frac{\beta}{\mu_i}\) for \(i = 1, 2\)

\(2\) \(\eta_1^+ < \eta_2^+\)

**Proof.** In the Appendix.

We shall now proceed to the explicit description of the manager’s optimal policies and value function for different choices of the pair of available strategies. By the result of part (2) of Lemma 1, since we must have \(\eta_1^+ < \eta_2^+\), there are only two different scenarios described by the ordering of the negative roots \(\eta_1^{-}, \eta_2^{-}\) of equations (7) that depend on the drift-volatility parameters characterizing the two strategies.

**Case 1:** \(\eta_1^{-} < \eta_2^{-} < \eta_1^{+} < \eta_2^{+}\)

We can show that the roots are distributed as above when (roughly) the difference between the drifts of the available strategies \(\mu_1 - \mu_2\) is large compared with the difference in the volatilities \(\sigma_1 - \sigma_2\).

The following proposition completely characterizes the optimal policy for the manager in this case.

**Proposition 2**

*The optimal policy for the manager is to choose strategy 1 throughout, i.e. asset substitution will never occur. Moreover, this is the unique optimal policy for the manager.*

**Proof.** In the Appendix.

\textsuperscript{15} This assumption avoids unnecessarily complicating the statements of several propositions.
The intuition for the above result is that even though the volatility of strategy 1 is higher than the volatility of strategy 2, its drift or expected return is much higher so that, due to the convexity of his compensation when the firm is solvent, it is optimal for the manager to choose strategy 1 always and never shift to strategy 2 even if the firm is close to the liquidation level and he risks losing his job.

**Case 2**: \( \eta^-_2 < \eta^-_1 < \eta^+_1 < \eta^+_2 \)

This corresponds (roughly) to the situation where the difference in the drifts of the strategies \( \mu_1 - \mu_2 \) is small compared with the difference in the volatilities \( \sigma_1 - \sigma_2 \).

We shall show that the manager’s unique optimal policies are to choose strategy 2 whenever the value of the state variable \( p \leq p_* \) and strategy 1 whenever \( p > p_* \) for some \( p_* \) with \( p_b \leq p_* < \infty \). In other words, the manager’s (stationary) optimal policies are to switch strategies whenever the value of the state variable crosses \( p_* \). In the case where \( p_* = p_b \), the optimal policy for the manager is to always choose strategy 1.

The intuition for this result is that, in this case, the difference in the volatilities of the two strategies outweighs the difference in their expected returns so that when the firm’s performance is mediocre, the manager would prefer to lower risk in order to preserve his job. However, when the firm is performing extremely well, the manager would prefer the strategy with higher expected return even though it has higher volatility due to the fact that the risk of the firm being liquidated and the manager losing his job is very low and his compensation structure is convex. We will provide precise analytical characterizations of the optimal policies. Since the arguments are rather involved we shall present the results in the form of three propositions, 4, 5, and 6. Before proceeding, we need to introduce some notation.

For each \( r \) with \( p_b \leq r < \infty \), let \( u_r \) be the value function of the policy (of the manager’s expected discounted compensation given by (3)) where the manager chooses strategy 2 for \( p \leq r \) and strategy 1 for \( p > r \). We can clearly distinguish two scenarios:

**A**: \( p_b \leq r \leq \frac{q}{\lambda} \)

In this case, we see from the results we have obtained thus far (see the proof of Proposition 2), that
where the subscripts indicate the explicit dependence on the switching point $r$.

B: $r \geq \frac{q}{\lambda}$

In this case, we see that

\[
\begin{align*}
u_r &= A_r p_r + B_r p_r^+ \frac{f}{\beta}; \quad p_b \leq p < r \\
&= C_r p_r + D_r p_r^+ \frac{f}{\beta}; \quad r \leq p < \frac{q}{\lambda} \\
&= E_r p_r + f - g q \frac{1}{\beta} + \frac{g \lambda p}{\beta - \mu}; \quad p \geq \frac{q}{\lambda} \\
u_r(p_b) &= 0
\end{align*}
\]

where the subscripts indicate the explicit dependence on the switching level $r$. We have retained the same variables for the coefficients defining the value functions in the two cases for the sake of notational brevity. The coefficients are uniquely determined by the condition that the value function is continuously differentiable for $p > p_b$ and continuous at $p_b$.

We can now state the following proposition that provides a necessary and sufficient condition for the unique optimal policy for the manager to choose strategy 1 throughout.

**Proposition 3**

*A necessary and sufficient condition for the choice of strategy 1 throughout to be the unique optimal policy for the manager is*

\[
L^2(u_{p_b}) + f + g (\lambda p - q)^+ \big|_{p=p_0+} \leq 0
\]
Note: $u_{p_b}$ denotes the value function of the policy of switching strategies at $p = p_b$, i.e. of choosing strategy 1 throughout so that the expression $L^2(u_{p_b}) + f + g(\lambda p - q)^+$ is a function of $p$. This function is evaluated at the point $p = p_b^+$, i.e. the limit as $p \to p_b$ in (10).

Proof. In the Appendix.

When condition (10) does not hold, we will show that there exists a switching point $p_\ast > p_b$ such that the policy of choosing strategy 2 for $p \leq p_\ast$ and strategy 1 for $p > p_\ast$ is optimal with $u_{p_\ast}$ defined as in (8) or (9) being the corresponding optimal value function.

We note from (8) and (9) that the value function $u_\ast$ is twice differentiable on $(p_b, \infty)$ except possibly at $p = r$. The basic idea in the proofs of the results that follow is to show the existence of a value $p_\ast$ such that the value function $u_{p_\ast}$ is twice differentiable at $p = p_\ast$ and hence on $(p_b, \infty)$. We will then show that the hypotheses of Proposition 1 are satisfied to conclude that the policy is therefore optimal.

Proposition 4

If

(11) $L^2(u_{p_b}) + f + g(\lambda p - q)^+ \big|_{p=p_b^+} > 0$

and

(12) $L^2(u_{q/\lambda}) + f + g(\lambda p - q)^+ \big|_{p=q/\lambda} \leq 0$

then there exists $p_\ast$ with $p_b < p_\ast \leq q/\lambda$ such the policy of choosing strategy 2 for $p \leq p_\ast$ and strategy 1 for $p > p_\ast$ is the unique optimal policy for the manager and $u_{p_\ast}$ defined in (8) is the corresponding optimal value function.

Proof. In the Appendix.

The result of the above proposition tells us that when conditions (11) and (12) hold it is optimal for the manager to switch strategies at $p_\ast \leq q/\lambda$, i.e. when the firm’s cash flows cannot

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16 This is the “super contact” condition discussed by Dumas [1991] and Mella-Barral and Perraudin [1997]
meet required debt payments. The following proposition shows that conditions (11) and (12) are both necessary and sufficient for asset substitution to occur at \( p, \leq q/\lambda \).

**Proposition 5**

*If condition (11) holds and*

\[
L^2(u_{q/\lambda}) + f + g(\lambda p - q)^+ |_{p=q/\lambda} > 0
\]

*then there exists \( p_* \) with \( \frac{q}{\lambda} < p_* < \infty \) such the policy of choosing strategy 2 when \( p_b < p \leq p_* \) and strategy 1 for \( p > p_* \) is the unique optimal policy for the manager.*

**Proof.** In the Appendix

Since the volatility of strategy 1 is greater than that of strategy 2 by definition, the results of **Propositions 3, 4** and **5** clearly imply that if it is ever optimal for the manager to switch strategies, he will always (roughly) choose the *low-volatility* strategy, i.e. strategy 2 close to the liquidation level and the *high-volatility* strategy, i.e. strategy 1, away from the liquidation level. As we shall see in the next section, the optimal policies for the firm are exactly the reverse, i.e. the firm will always choose the *high-volatility* strategy close to the liquidation level and the *low-volatility* strategy away from the liquidation level. This is exactly the source of managerial flexibility that is the primary focus of this paper. The results of **Propositions 3, 4 and 5** combine to completely characterize the manager’s optimal policies. In particular, we see that is never optimal for the manager to choose strategy 2, i.e. the low volatility strategy throughout if he is compensated at least partly with executive options, i.e. \( g > 0 \). We provide necessary and sufficient conditions for asset substitution to be optimal and for risk-shifting to occur above or below the illiquidity threshold \( \frac{q}{\lambda} \).

**5. Optimal Policies for the Firm**

In this section, we shall explicitly derive the optimal policies for the firm, i.e. the policies that maximize the *market value* of the firm when the firm can hypothetically contract for managerial behavior *ex ante*. We show that
the optimal policies are EITHER to choose strategy 2, i.e. the low volatility strategy throughout OR to switch from the high volatility strategy, i.e. strategy 1 at low asset values to the low volatility strategy, i.e. strategy 2 at high asset values.

- We provide analytical necessary and sufficient conditions for each of the cases above.

We consider the general situation where debt has tax benefits and there are costs associated with liquidation. Liquidation is assumed to occur *exogenously* and debt is serviced strategically as discussed in Section 1. Debt service represents a redistribution of wealth between shareholders and creditors that does not affect overall firm cash flows. The tax rate is denoted by \( \tau \) and the exogenous liquidation level by \( p_b \) which may be a function of the coupon rate \( \delta \). We assume throughout that \( p_b \leq \frac{\delta}{\lambda} \), i.e. the firm will never be liquidated when it is profitable (net of interest payments).

The cash flows per unit time associated with the firm after debt is in place and when the firm is in operation are therefore given by

\[
C(P(t)) = \lambda P(t) + \tau \delta; \quad P(t) \geq \delta / \lambda \\
= \lambda P(t) + \tau \lambda P(t); \quad p_b \leq P(t) \leq \delta / \lambda
\]  

(14)

Since our goal in this section is the maximization of the *market value* of the firm, we work under the *risk neutral measure* under which the drifts of both strategies are equal to \( r - \delta \) where \( r \) is the risk free rate and \( \delta \) is the payout rate for the state variable \( P(.) \) that we have assumed to be the price process of some traded asset in the market. Therefore, the state variable \( P(.) \) evolves as follows under the *risk neutral measure*:

\[
dP(t) = (r - \delta)P(t)dt + \sigma_1 P(t)dB^1(t); \quad \text{Strategy 1} \\
dP(t) = (r - \delta)P(t)dt + \sigma_2 P(t)dB^2(t); \quad \text{Strategy 2}
\]  

(15)

Since the costs of liquidation are proportional to the value of the unlevered assets of the firm, it is clearly never optimal for an all-equity financed or unlevered firm to be liquidated at any nonzero value of the state variable \( P(.) \). Therefore, we easily see from (14) and (15) with \( p_b = 0 \) and \( \tau = 0 \) (there are no tax advantages) that the value of the unlevered firm when the state variable \( P(.) = p \) is...
given by $V(p) = \frac{\lambda p}{\delta}$. Therefore, in the presence of debt, the value of the firm at the exogenous liquidation level $p_b$ is given by $V(p_b) = (1 - \alpha) \frac{\lambda p_b}{\delta}$ for some $\alpha < 1$.

The goal is to maximize the market value of the firm that is given by

$$V_t(p) = E_t\left[ \int_0^{\tau_d} \exp(-rt)C(P(t))dt + (1 - \alpha) \frac{\lambda p_b}{\delta} \right]$$

under the switching policy $\Gamma$.

As in (4), (5), (6) we can introduce the formal Hamilton-Jacobi-Bellman equation associated with the firm’s optimization problem

$$-ru + \sup_{r=1,2} \left[ (r - \delta) p u_p + \frac{1}{2} \sigma^2_p p^2 u_{pp} \right] + C(p) = 0, p \in (p_b, \infty)$$

and the generators of the strategies available to the firm

$$L^1(u) = -ru + (r - \delta) p u_p + \frac{1}{2} \sigma^2_p p^2 u_{pp},$$

$$L^2(u) = -ru + (r - \delta) p u_p + \frac{1}{2} \sigma^2_p p^2 u_{pp}$$

Analogous to (7), we obtain the following pair of characteristic equations of the generators above:

$$\frac{1}{2} \sigma^2_1 x^2 + (r - \delta - \frac{1}{2} \sigma^2_1) x - r = 0 \quad \text{with roots } \rho_1^+, \rho_1^-$$

$$\frac{1}{2} \sigma^2_2 x^2 + (r - \delta - \frac{1}{2} \sigma^2_2) x - r = 0 \quad \text{with roots } \rho_2^+, \rho_2^-$$

By the results of Lemma 1, we see that

$$\rho_2^- < \rho_1^- < \rho_1^+ < \rho_2^+ \quad \text{if } \sigma_1 > \sigma_2$$

We can now state the following proposition that completely specifies the optimal policies for the firm.

**Proposition 6**

With exogenous liquidation at the level $p_b \leq \delta / \lambda$, there exists $p^*$, with $p_b \leq p^* < \delta / \lambda$ such that the unique optimal policy for the firm is to choose the high volatility strategy, i.e. strategy 1 for $p \leq p^*$.
and the low volatility strategy, i.e. strategy 2 for \( p \geq p^* \). We may have \( p^* = p_b \) in which case it is optimal for the firm to choose the low volatility strategy throughout.

*Proof.* In the Appendix.

By the result of the above proposition, we see that it is not always optimal for a firm to increase volatility close to liquidation. The proof of the proposition provides a very precise necessary and sufficient condition under which it is optimal for the firm to increase risk close to bankruptcy. Moreover, the risk-shifting threshold \( p^* \) always lies below the threshold \( \hat{\sigma} / \lambda \) where the firm’s cash flows are just sufficient to meet interest payments.

Intuitively, the reasons for the conflict of interest between manager and the firm is that the firm’s goal is to maximize its market value, i.e. its expected discounted cash flows in a risk neutral world and that the firm’s cash flows described by (14) are concave in the value of the state variable \( P(.) \). This is in contrast with the fact that the manager’s goal is to maximize his expected discounted cash flows in the real world and that his compensation is convex in the value of the state variable \( P(.) \).

4. Optimal Managerial Compensation, Capital Structure and Agency Costs

We can use the results of the previous sections to explicitly derive the agency costs of managerial flexibility. By comparing the hypothetical situation where the manager’s policies can be contracted for ex ante, (i.e. before debt is in place) and the actual situation where the manager’s interests may conflict with those of the firm, i.e. he chooses his policies ex post given his compensation structure, (i.e. after debt is in place), we can obtain a measure of the agency costs of debt due to managerial flexibility that is inspired by Leland [1998]. Moreover, the manager’s optimal compensation defined by the parameters \( f, g, q \) defined in Sections 2 and 3 may be derived endogenously. Specifically, the optimal parameters \( f, g, q \) are those that maximize the shareholders’ value of equity. We will elaborate on this later in the section.
From Section 1, we recall our assumption that debt is serviced entirely by shareholders until the state variable \( P(.) \) reaches a level \( p_e \) where the value of equity is zero. When \( p_b \leq P(.) \leq p_e \), bondholders make concessions by accepting all cash flows from the firm's operations. The actual form of debt service, however, does not affect the value of the firm since firm-related cash flows are unchanged by the costless redistribution of wealth between shareholders and bondholders.

The Ex-Post Value Functions, Managerial Compensation and Optimal Capital Structure

In Section 2, we have shown that given an exogenously specified compensation structure described by the parameters \( f, g, q \), the optimal policies for the manager are always to choose strategy 2 for \( p \leq p_* \) and strategy 1 for \( p > p_* \) for some \( p_* \) with \( p_b \leq p_* < \infty \). In particular, when \( p_b = p_* \), the optimal policy is to choose strategy 1 throughout. We can now use the results of the previous section to obtain the value function of the firm corresponding to this policy. We clearly have three different possibilities:

Case 1: \( p_* = p_b \)

In this case, the value function of the firm is given by

\[
v(p) = E p^{\delta_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad p \geq \frac{\delta}{\lambda}
\]

\[
= C p^{\delta_i} + D p^{\delta_i} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta}; \quad p_b < p \leq \frac{\delta}{\lambda}
\]

\[
v(p_b) = (1 - \alpha) \frac{\lambda p_b}{\delta}
\]

The determination of the value of the firm’s debt is complicated by the form of debt service during financial distress. If \( p_e \) is the (endogenously determined) level at which the value of equity falls to zero, then the value of debt is given by

\[
d(p) = J p^{\delta_i} + \frac{\delta}{p}; \quad p \geq p_e
\]

\[
= v(p); \quad p < p_e
\]

Note: The value of equity \( e(.) = v(.) - d(.) \)
Case 2: \( p_b < p_s \leq \frac{\delta}{\lambda} \)

In this case, the value function of the firm is given by

\[
v(p) = Ep^{\rho_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad p \geq \frac{\delta}{\lambda}
\]

\[= Cp^{\rho_i} + Dp^{\rho_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad p_s < p \leq \frac{\delta}{\lambda}
\]

\[
(18)
\]

\[= Ap^{\rho_i} + Bp^{\rho_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad p_b < p \leq p_s
\]

\[v(p_b) = (1 - \alpha) \frac{\lambda p_b}{\delta}
\]

If \( p_e < p_s \), the value of the firm’s debt is given by

\[
d(p) = Jp^{\rho_i} + \frac{\delta}{r}; \quad p > p_s
\]

\[= Fp^{\rho_i} + Gp^{\rho_i} + \frac{\delta}{r}; \quad p_e < p \leq p_s
\]

\[d(p) = v(p); \quad p \leq p_e
\]

and if \( p_e \geq p_s \), it is given by

\[
d(p) = Jp^{\rho_i} + \frac{\delta}{r}; \quad p > p_e
\]

\[= Fp^{\rho_i} + Gp^{\rho_i} + \frac{\delta}{r}; \quad p_s < p \leq p_e
\]

\[d(p) = v(p); \quad p \leq p_e
\]

Case 3: \( \frac{\delta}{\lambda} < p_s < \infty \)

In this case, the firm’s value function is given by

\[
v(p) = Ep^{\rho_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad p \geq p_s
\]

\[= Cp^{\rho_i} + Dp^{\rho_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad \frac{\delta}{\lambda} < p < p_s
\]

\[
(19)
\]

\[= Ap^{\rho_i} + Bp^{\rho_i} + \frac{\lambda p}{\delta} + \frac{\tau \delta}{r}; \quad p_b < p \leq \frac{\delta}{\lambda}
\]
with \( v(p_b) = (1 - \alpha) \frac{\lambda p_b}{\delta} \)

and the value of its debt (since \( p_e \leq \frac{\vartheta}{\lambda} < p_\infty \)) is given by

\[
d(p) = Jp^\rho + \frac{\vartheta}{r}; p \geq p_\infty
\]

\[
= Hp^\rho + Ip^\rho + \frac{\vartheta}{r}; p_e < p < p_\infty
\]

\[
d(p) = v(p); p \leq p_e
\]

We have obtained precise necessary and sufficient conditions for each of the 3 cases above in Section 2. The coefficients in the expressions above are determined by the conditions that the value function and value of debt are continuous for \( p \geq p_b \) and differentiable for \( p > p_b \). This implies that the level \( p_e \) is endogenously determined by the condition that the value of equity, i.e. the difference between the value of the firm and the value of debt, is zero at \( p_e \) and its first derivative is also zero (the smooth pasting condition). If no such point exists, i.e. the value of equity is positive for all \( p > p_b \), then we set \( p_e = p_b \) in the above expressions. Since shareholders are residual claimants, the boundary condition on the value of debt at the liquidation level \( p_b \) is given by

\[
d(p_b) = \min\left( \frac{\vartheta}{r}, (1 - \alpha) \frac{\lambda p_b}{\delta} \right)
\]

since \( \frac{\vartheta}{r} \) is the outstanding principal payment due to bondholders.

**Optimal Managerial Compensation**

The manager’s compensation (defined by the parameters \( f, g, q \)) may be endogenously derived ex post as that which maximizes shareholders’ value of equity \( e(.) \). Specifically, the “optimal” parameters \( f^*, g^*, q^* \) are obtained as follows:

\[
(f^*, g^*, q^*) = \arg \max_{(f,g,q)} e(p_0)
\]

In the above, \( p_0 \) is the initial value of the state variable \( P(.) \) that defines the cash flows of the firm. The value of equity \( e(p_0) = v(p_0) - d(p_0) \) may be obtained from equations (17)-(20). Since the
parameters \( f^*, g^*, q^* \) are determined ex-post, i.e. after debt is in place, they depend in general on the coupon rate \( \vartheta \). Since the manager’s expected discounted compensation described by (3) is homogenous of degree one in the parameters \((f, g)\), his optimal policies are unchanged if \((f, g)\) are scaled by the same factor. Our assumption that the manager’s compensation is negligible compared with the firm’s cash flows therefore implies that we may without loss of generality assume that \( g = 1 \). Therefore, equation (21) may be replaced with

\[
(f^*, q^*) = \arg \max_{(f, q)} e(p_0)
\]

**Optimal Ex-Post Leverage**

From (22), we obtain the optimal option compensation structure of the manager for each value of the coupon rate \( \vartheta \). We can therefore use equation (17), (18) or (19) to derive the value of the firm when the manager is compensated optimally at the coupon rate \( \vartheta \). Let us denote this value by \( v^*(p_0, \vartheta) \) indicating the explicit dependence on the initial value \( p_0 \) of the unlevered assets of the firm and the coupon rate \( \vartheta \). The optimal ex-post leverage and hence capital structure of the firm is then obtained by maximizing firm value \( v^*(p_0, \vartheta) \) as a function of the coupon rate \( \vartheta \).

**The Ex-Ante Value Functions and Optimal Capital Structure**

In order to obtain the agency costs due to managerial flexibility, we use the procedure suggested by Leland [1998] to investigate the hypothetical situation where managerial policy can be contracted for ex ante, i.e. the firm can choose policies so as to maximize firm value. In this case, we can directly apply the results of Section 3 to write down the value functions of the firm. The optimal policies of the firm are to choose strategy 1, i.e. the high volatility strategy for \( p \leq p^* \) and the low volatility strategy, i.e. strategy 2 for \( p > p^* \) where \( p_b \leq p^* < \frac{\vartheta}{\lambda} \). As we have discussed in Section 3, there are only two possibilities:

**Case 1:** \( p^* = p_b \)

The firm’s value function is given by
\[ v(p) = Ep^\frac{\delta}{\delta} + \frac{\lambda p}{\delta} + \frac{\tau \theta}{\delta} r; p \geq \frac{\rho}{\lambda} \]

\[ = Cp^\frac{\delta}{\delta} + Dp^\frac{\delta}{\delta} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta} r; p_b < p \leq \frac{\rho}{\lambda} \]

\[ v(p_b) = (1 - \alpha) \frac{\lambda p_b}{\delta} \]

and the value of its debt in terms of the (endogenously determined) level \( p_e \) at which the value of equity falls to zero is given by

\[ d(p) = Jp^\frac{\delta}{\delta} + \frac{\rho}{\delta} r; p \geq p_e \]

\[ = v(p); p < p_e \]

**Case 2:** \( p_b < p^* < \frac{\rho}{\lambda} \)

The firm’s value function is given by

\[ v(p) = Ep^\frac{\delta}{\delta} + \frac{\lambda p}{\delta} + \frac{\tau \theta}{\delta} r; p \geq \frac{\rho}{\lambda} \]

\[ = Cp^\frac{\delta}{\delta} + Dp^\frac{\delta}{\delta} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta} r; p^* < p \leq \frac{\rho}{\lambda} \]

\[ = Ap^\frac{\delta}{\delta} + Bp^\frac{\delta}{\delta} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta} r; p_b < p \leq p^* \]

\[ v(p_b) = (1 - \alpha) \frac{\lambda p_b}{\delta} \]

and the value of its debt if \( p_e < p^* \) is given by

\[ d(p) = Jp^\frac{\delta}{\delta} + \frac{\rho}{\delta} r; p \geq p^* \]

\[ = Hp^\frac{\delta}{\delta} + Ip^\frac{\delta}{\delta} + \frac{\rho}{\delta} r; p_e < p \leq p^* \]

\[ = v(p); p \leq p_e \]

and if \( p_e > p^* \) is given by
$d(p) = J p^\gamma + \frac{\vartheta}{r}; p \geq p_e$

$= H p^\gamma + J p^\gamma + \frac{\vartheta}{r}; p < p \leq p^*$

$= v(p); p \leq p_e$

We obtain the optimal ex-ante leverage by maximizing the firm’s value function as a function of the coupon rate $\vartheta$. The coefficients and the endogenous level $p_e$ in all cases above are determined by the conditions that the value function and the value of debt be continuous for $p \geq p_b$ and differentiable for $p > p_b$. If $p_e$ does not exist, i.e. the value of equity is positive for all $p > p_b$, then we set $p_e = p_b$ in the above expressions and set $d(p_b) = \min(\frac{\vartheta}{r}, (1-\alpha)^{-\frac{\lambda p_b}{\delta})).$

If $v_{\text{post}}(p_0), v_{\text{ante}}(p_0)$ denote the optimal value functions (i.e. value functions at the optimal leverage) ex post and ex ante respectively where $p_0$ is the initial value of the state variable, then as in Leland [1998], we have

$$\text{(23) Agency Costs} = v_{\text{ante}}(p_0) - v_{\text{post}}(p_0)$$

Equation (23) therefore quantifies the agency costs of managerial flexibility incorporating the fact that managerial compensation is endogenously determined ex-post by shareholders who maximize the value of their equity.

In the next section, we present the results of several numerical simulations we have carried out that allow us to evaluate the significance of managerial flexibility as a determinant of capital structure and the valuation of corporate debt.

5. Numerical Simulations

In all the numerical simulations whose results we present, we have assumed that the exogenous liquidation level $p_b$ is proportional to the coupon rate $\vartheta$. More precisely, we assume that

$$\text{(24) } p_b = \varepsilon \frac{\vartheta}{\lambda} \text{ where } 0 < \varepsilon \leq 1$$
Recall that $\delta / \lambda$ is the “illiquidity threshold”, i.e. it is the value of the state variable below which the cash flows from the firm’s operations are not sufficient to meet interest payments. For example, the liquidation level would have the form (24) if liquidation were triggered at the level where the value of unlevered assets of the firm $\frac{\lambda d}{\delta}$ falls below the outstanding debt principal $\frac{\delta}{r}$.

A. Agency Costs due to Managerial Flexibility

We have numerically implemented the results of the previous section to endogenously derive the manager’s optimal compensation structure and then evaluate the optimal ex ante and ex post value functions of the firm in order to derive the agency costs from (23). Tables 1 and 2 present our results for different choices of the parameter values of our model. As is clear from the tables, the agency costs of debt due to managerial flexibility are, in general, very significant in comparison with the tax advantages of debt. The difference in the leverage the firm may take on ex ante and ex post is also very significant for reasonable choices of parameter values. When contrasted with the results of Leland [1998], these results demonstrate that managerial flexibility is a far more significant determinant of the optimal capital structure of the firm than asset substitution driven by shareholders’ interests.

We endogenously derive the optimal compensation of the manager as described in (22) and (32). We have also displayed the optimal risk-shifting points for the manager and the optimal risk-shifting points for the firm at the optimal leverage, and the exogenous liquidation levels. Since the value of the unlevered assets of the firm and the value of the state variable are in one-one correspondence with each other, we describe these points in terms of the value of the unlevered assets at these points. The initial value of the unlevered assets is always assumed to be 100. We notice that in both cases, the manager’s options are in the money. Additionally, it is interesting note that the linear compensation structure is not optimal since the executive options have a nonzero strike price. However, in the second example the optimal compensation structure is “almost linear” since the strike price of the executive options is $1.25 that is small compared with the initial value of the unlevered assets $100. We display the credit spread at the optimal ex post leverage that is the difference between the interest rate the firm pays and the interest rate it would pay if its strategy were risk free.
B. Variation of Agency Costs with the Drifts of the Strategies

It is also interesting to investigate how agency costs vary with the drifts of the available strategies. Figure 1 displays the variation of agency costs with the drift of strategy 2 assuming that the drift of strategy 1 and the volatilities of both strategies are kept fixed. The values of the other parameters are the same as in the case of Table 1. This investigation also allows us to easily quantify the agency costs of debt due to asymmetric information between the manager and investors regarding the expected returns of the available strategies. If the investors only know the risks of the available strategies, but not their expected returns, then both equity and debt would be valued assuming the worst-case scenario. Figure 1 shows that the agency costs when investors do not know the drift of strategy 2 would be just under 3.5%.

C. Variation of Optimal Ex Ante and Ex Post Leverage with Drifts

Figure 2 displays the variation of the optimal ex ante and ex post leverage of the firm with the drift of strategy 2 with the values of the other parameters being the same as in the case of Figure 1. The ex ante leverage does not vary with the drift of strategy 2 since the ex ante optimal policies of the firm do not depend on the drifts of the strategies as explained in Section 2 since firm value maximization is under the risk neutral measure. We notice that the optimal ex post leverage increases with the drift of strategy 2 when the drift of strategy 1 is kept fixed. If strategy 1 is interpreted as a “high growth” strategy, then this observation implies that if the difference between the firm’s higher growth and conservative strategies becomes less significant, it takes on higher leverage levels. This is consistent with the empirical results reported in Mehran [1992] that firms without significant growth opportunities take on higher leverage levels compared with firms that do.

D. Variation of Agency Costs with the Volatilities of the Strategies

Figure 3 displays the variation of agency costs with the volatility of strategy 2 assuming that the volatility of strategy 1 and the drifts of both strategies are kept fixed. We notice that the agency costs go to zero as the volatilities of the strategies converge since the strategies are then indistinguishable from the standpoint of the firm so that the optimal ex ante and ex post value functions would converge.
E. Variation of Optimal Ex Ante and Ex Post Leverage with Volatility of Strategy 2

Figure 4 displays the variation of the optimal ex ante and ex post leverage of the firm with the volatility of strategy 2. We notice that both the optimal ex ante and ex ante leverages decrease with the volatility of strategy 2 ceteris paribus, i.e. as the difference of the volatilities of the firm’s possible strategies decreases. When the volatilities of the two strategies converge, it implies that the firm does not have a significantly less risky conservative strategy. Therefore, the firm is on an average “riskier” and therefore takes on lower leverage levels.

From Figures 2 and 4, we see that the optimal ex post leverage, i.e. the optimal leverage of the firm in the presence of managerial flexibility varies between 10% and 30% that corresponds quite well with average leverage levels observed in the market.

6. Conclusions

In this paper, we examined the significance of managerial flexibility as a determinant of the capital structure of a firm. The manager, who is assumed to be risk-neutral, may dynamically switch between two strategies with different risks and expected returns and also bears significant personal costs due to liquidation of the firm.

We demonstrated that the manager’s unique optimal policies are to choose the low risk (and low expected return strategy) whenever the unlevered asset value of the firm is below an endogenously derived threshold and the high risk (and high expected return strategy) whenever the unlevered asset value of the firm is above the threshold and presented precise necessary and sufficient conditions for the location of the risk-shifting threshold.

We then investigated the optimal policies for the firm that can hypothetically contract for managerial behavior ex ante (i.e. before debt is in place) and demonstrated that the optimal policies for the firm are to choose the high risk strategy whenever the unlevered asset value is below an endogenously derived threshold and the low risk strategy whenever the unlevered asset value is above the threshold. We demonstrated that the threshold is always below the illiquidity threshold, i.e. the firm will switch strategies when it is unprofitable net of contractual debt payments.

The fundamental dichotomy between the optimal behavior of the manager and that of the firm is the principle contributor to the agency costs of debt due to managerial flexibility. We
demonstrated the significance of managerial flexibility as a determinant of optimal capital structure through several numerical simulations.

Our analytical characterizations of optimal managerial and firm behavior provide insights into the problem of designing optimal compensation contracts for the manager that could analyze managerial incentives with those of the firm and thus hopefully eliminate managerial flexibility.

Limitations and Extensions

In this paper, we have examined the situation where the firm issues perpetual debt. This is an idealization of reality that has been imposed for analytical tractability. Although this is a good approximation for a firm issuing long-term debt, it is clearly not valid in the modeling of medium of short-term debt. It would be interesting to examine the influence of managerial asset substitution on the capital structure of a firm issuing medium or short-term debt. It is quite likely that the model would not be amenable to analytical results, but one could, in principle, adopt a numerical approach similar to that of Anderson and Sundaresan [1996] to investigate this problem. Such an analysis would contribute to the important goal of studying the optimal debt structure of a firm in the presence of managerial flexibility.

APPENDIX

Proof of Lemma 1

(1) We notice that

\[ 0 > \mu_i - \beta = \frac{1}{2} \sigma_i^2 (1)^2 + (\mu_i - \frac{1}{2} \sigma_i^2)(1) - \beta = \frac{1}{2} \sigma_i^2 (1 - \eta_i^+)(1 - \eta_i^-) \quad \text{for } i = 1, 2 \]

where the first inequality above follows from our hypothesis that \( \beta > \mu_i \) and the last equality follows from the definitions of the roots of equations (10). It follows from the above that we must have \( \eta_i^- < 1 < \eta_i^+ \). Next, we note that

\[ \frac{1}{2} \sigma_i^2 \left( \frac{\beta}{\mu_i} - \eta_i^+ \right) \left( \frac{\beta}{\mu_i} - \eta_i^- \right) = \frac{1}{2} \sigma_i^2 \left( \frac{\beta}{\mu_i} \right)^2 + (\mu_i - \frac{1}{2} \sigma_i^2) \frac{\beta}{\mu_i} - \beta = \frac{1}{2} \sigma_i^2 \left( \frac{\beta^2}{\mu_i^2} - \frac{\beta}{\mu_i} \right) > 0 \]

since \( \frac{\beta}{\mu_i} > 1 \). It follows from the above that \( \eta_i^- < \frac{\beta}{\mu_i} \).
(2) We have
\[ \frac{1}{2} \sigma_1^2 \eta_2^2 + (\mu - \frac{1}{2} \sigma_1^2) \eta_2 - \beta = \frac{1}{2} \sigma_1^2 (\eta_2^2 - \eta_2^*) + \mu \eta_2^* - \beta > \frac{1}{2} \sigma_2^2 (\eta_2^2 - \eta_2^*) + \mu \eta_2^* - \beta = 0 \] since \( \mu_1 > \mu_2, \sigma_1 > \sigma_2, \eta_2^* > 1 \). Therefore,
\[ \frac{1}{2} \sigma_1^2 (\eta_2^2 - \eta_2^*) (\eta_2^* - \eta_2^-) = \frac{1}{2} \sigma_1^2 \eta_2^2 + (\mu - \frac{1}{2} \sigma_1^2) \eta_2^* - \beta > 0 \]
It follows that \( \eta_2^* \) must be greater than \( \eta_1^*, \eta_1^- \), i.e. \( \eta_1^+ < \eta_2^+ \). This completes the proof of the lemma.

\[ \diamond \]

**Proof of Proposition 2**

The proof proceeds by the explicit construction of a function \( u \) satisfying the hypotheses of Proposition 1 thereby implying that it is the value function. In the process, we shall also show that the optimal policy for the manager is to choose strategy 1 throughout. We distinguish two regions:

Region I: \( p_b \leq p < q/\lambda \)

Since our hypothesized optimal policy is to choose strategy 1, (5) implies that the value function \( u \) must satisfy

\[(A1) \quad L^1(u) = -\beta u + \mu_1 p u_p + \frac{1}{2} \sigma_1^2 p^2 u_{pp} + f = 0.\]

It is well known that any solution to (A1) has the general form \( u = Ap^\eta_i + Bp^\eta_i + \frac{f}{\beta} \)

By the boundary condition on \( u \) at \( p = p_b \), we have

\[(A2) \quad Ap_b^\eta_i + Bp_b^\eta_i + \frac{f}{\beta} = 0 \]

Therefore,

\[(A3) \quad u_i = Ap^\eta_i + Bp^\eta_i + \frac{f}{\beta}, \quad u_i(p_b) = 0 \]

where the subscript on \( u \) denotes that this is the value function in region I.

Region II: \( p > q/\lambda \)

In this case, the value function must satisfy

\[(A4) \quad L^1(u) + f + g(\lambda p - q) = 0 \]
The general solution to the above equation must be of the form
\[ u = C p^{\eta_i} + D p^{\eta_i} + \frac{\tilde{f}}{\beta} + \frac{\tilde{g} \lambda p}{\beta - \mu_i} - \frac{g q}{\beta} \]

For very large values of \( p \), the manager is almost certain to obtain cash flows at the rate \( f + \tilde{g} \lambda p - g q \) so that we must have \( C = 0 \) in the equation above. Therefore,
\[ u_{ii} = D p^{\eta_i} + \frac{\tilde{f}}{\beta} + \frac{\tilde{g} \lambda p}{\beta - \mu_i} - \frac{g q}{\beta} \]

We define our hypothesized value function \( u \) to be equal to \( u_i \) in region \( I \) and equal to \( u_{ii} \) in region \( II \). For the function \( u \) to be \( C^1 \), we match its value and its first derivative at the boundary \( p = q / \lambda \) thereby obtaining
\[ \frac{A(q/\lambda)^{\eta_i}}{\beta - \mu_i} = \frac{(1 - \eta_i^2)g q}{\beta (\eta_i^2 - \eta_i)} + \frac{\eta_i g q}{\beta (\eta_i^2 - \eta_i)} = \frac{(\beta - \mu_i \eta_i^2) g q}{\beta (\beta - \mu_i (\eta_i^2 - \eta_i))} > 0 \]

since \( \eta_i < 0 \). Hence, we see that
\[ A > 0 \]

Since \( u_i(p_b) = 0 \), we must have
\[ A(p_b)^{\eta_i} + B(p_b)^{\eta_i} = -\frac{f}{\beta} \]

Since \( A > 0 \) from (A7), we must have
\[ B < 0 \]

By the result of Proposition 1, \( u \) is the value function if and only if
\[ L^2(u_i) + f + g(\lambda p - q)^+ \leq 0 \]
\[ L^2(u_{ii}) + f + g(\lambda p - q)^+ \leq 0 \]

We see that
\[ L^2(u_i) + f + g(\lambda p - q)^+ = L^2(\frac{A p^{\eta_i} + B p^{\eta_i} + \frac{f}{\beta}}{\beta}) + f + 0 \]
\[ = A p^{\eta_i} \left( \frac{1}{2} \sigma_2^2 (\eta_1^2)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2) \eta_1^2 - \beta \right) + B p^{\eta_i} \left( \frac{1}{2} \sigma_2^2 (\eta_1^2)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2) \eta_1^2 - \beta \right) \]
\[ = A p^{\eta_i} \frac{1}{2} \sigma_2^2 (\eta_1^2 - \eta_2^2)(\eta_1^2 - \eta_2^2) + B p^{\eta_i} \frac{1}{2} \sigma_2^2 (\eta_1^2 - \eta_2^2)(\eta_1^2 - \eta_2^2) \]
where the last equality follows from the fact that $\eta^+_2, \eta^-_2$ are the roots of the second equation in (7).

Since $\eta^-_1 < \eta^-_2 < \eta^+_1 < \eta^+_2$ by hypothesis and $A > 0, B < 0$ from (A7) and (A8), we easily see that

(A11) \[ L^2(u_t) + f + g(\lambda p - q)^+ < 0 \]

By construction, $u$ is twice differentiable at $p = \frac{q}{\lambda}$ since the value and first derivatives of $u_t, u_{tt}$ are equal at $p = \frac{q}{\lambda}$ and

\[ L^1(u_t) + f + g(\lambda p - q)^+ \Big|_{p=q/\lambda} = L^1(u_{tt}) + f + g(\lambda p - q)^+ \Big|_{p=q/\lambda} = 0 \]

by construction. Therefore, (A11) clearly implies that

(A12) \[ L^2(u_{tt}) + f + g(\lambda p - q)^+ \Big|_{p=q/\lambda} < 0 \]

We now note from (A5) that

(A13) \[ L^2(u_{tt}) + f + g(\lambda p - q)^+ = L^2(Dp^{n_0} + \frac{q - gq}{\beta} + \frac{g\lambda p}{(\beta - \mu_1)}) + f + g(\lambda p - q)^+ \]

\[ = Dp^{n_0} \frac{1}{2} \sigma^2 (\eta^-_1 - \eta^+_2)(\eta^-_1 - \eta^-_2) + \frac{g\lambda p (\mu_2 - \mu_1)}{(\beta - \mu_1)} \]

where the second equality above is obtained as in (A10). Since $\eta^-_1 < \eta^-_2 < \eta^+_1 < \eta^+_2$ and $\mu_2 < \mu_1$, we see from the above that if $D \leq 0$, $L^2(u_{tt}) + f + g(\lambda p - q)^+ < 0$.

On the other hand, if $D > 0$, the first term in the last expression in (A13) is positive. However, we note that in this case, $Dp^{n_0} \frac{1}{2} \sigma^2 (\eta^-_1 - \eta^+_2)(\eta^-_1 - \eta^-_2) + \frac{g\lambda p (\mu_2 - \mu_1)}{(\beta - \mu_1)}$ is a decreasing function of $p$. Since $L^2(u_{tt}) + f + g(\lambda p - q)^+ \Big|_{p=q/\lambda} < 0$ from (A12), we conclude that

$L^2(u_{tt}) + f + g(\lambda p - q)^+ < 0$ for $p \geq q/\lambda$ and therefore in region II. It follows that

(A14) \[ L^2(u_{tt}) + f + g(\lambda p - q)^+ < 0 \]

From (A11) and (A14) we see that the function $u$ satisfies the hypotheses of Proposition 1 and is therefore the value function of the manager's optimization problem. Moreover, the fact that the inequalities (A11), (A14) are strict implies (from standard programming arguments\(^\text{17}\)) that the policy of choosing strategy 1 throughout is the unique optimal policy for the manager. This completes the proof of the proposition. ♦

\(^\text{17}\) These are available from the author upon request.
Proof of Proposition 3

We shall only prove the sufficiency of condition (10). The necessity follows directly from the results of Propositions 4 and 5. By (8) we have

\[ u_{p_{b}} = C_{p_{b}} p^{n_{i}} + D_{p_{b}} p^{n_{i}} + \frac{f}{\beta} ; p_{b} \leq p < \frac{q}{\lambda} \]

\[ = E_{p_{b}} p^{n_{i}} + \frac{f - gq}{\beta} + \frac{g\lambda p}{\beta - \mu_{1}} ; p \geq \frac{q}{\lambda} \]

(A15)

\[ u_{p_{b}} (p_{b}) = 0 \]

By the matching of the value and first derivative at \( p = q/\lambda \), we obtain as in (A6) in the proof of Proposition 2 that

\[ C_{p_{b}} (q/\lambda)^{n_{i}} = \frac{(1-\eta_{1})gq}{(\beta - \mu_{1})(\eta_{i}^{+} - \eta_{i}^{-})} + \frac{\eta_{1}^{-} gq}{\beta (\eta_{i}^{+} - \eta_{i}^{-})} = \frac{(\beta - \mu_{1}\eta_{i}^{+})gq}{\beta (\beta - \mu_{1})(\eta_{i}^{+} - \eta_{i}^{-})} > 0 \]

(A16)

By the condition \( u_{p_{b}} (p_{b}) = 0 \), we see that \( D_{p_{b}} < 0 \). Therefore,

(A17) \[ C_{p_{b}} > 0, D_{p_{b}} < 0 \]

We now note that for \( p_{b} \leq p < q/\lambda \),

\[ L^{2}(u_{p_{b}}) + f + g(\lambda p - q)^{+} = L^{2}(u_{p_{b}}) + f \]

\[ = C_{p_{b}} p^{n_{i}} \frac{1}{2} \sigma_{2}^{2} (\eta_{i}^{+} - \eta_{i}^{-})(\eta_{i}^{+} - \eta_{i}^{-}) + D_{p_{b}} p^{n_{i}} \frac{1}{2} \sigma_{2}^{2} (\eta_{i}^{-} - \eta_{1}^{-})(\eta_{i}^{-} - \eta_{i}^{+}) \]

(A18)

Since \( \eta_{i}^{+} < \eta_{i}^{-} < \eta_{i}^{-} < \eta_{i}^{+} \) by hypothesis, we see from (A17) that the first term in the last expression above is negative and the second term is positive. Therefore, the expression is a decreasing function of \( p \) for \( p_{b} \leq p \leq q/\lambda \). It therefore follows from hypothesis (10) of the proposition that

(A19) \[ L^{2}(u_{p_{b}}) + f + g(\lambda p - q)^{+} < 0 \] for \( p_{b} \leq p \leq q/\lambda \).

For \( p \geq q/\lambda \),

\[ L^{2}(u_{p_{b}}) + f + g(\lambda p - q)^{+} = \]

\[ E_{p_{b}} p^{n_{i}} \frac{1}{2} \sigma_{2}^{2} (\eta_{i}^{-} - \eta_{2}^{-})(\eta_{i}^{-} - \eta_{2}^{+}) + \frac{g\lambda p(\mu_{2} - \mu_{1})}{\beta - \mu_{1}} \]

(A20)

If \( E_{p_{b}} \geq 0 \), we easily see from the fact that \( \mu_{2} < \mu_{1} \), and \( \eta_{i}^{+} < \eta_{i}^{-} < \eta_{i}^{+} < \eta_{2}^{+} \) that both terms in the second expression above are negative. Therefore,
\[ L^2(u_{p_b}) + f + g(\lambda p - q)^+ \text{ for } p \geq q/\lambda. \]

On the other hand, if \( E_{p_b} < 0 \), the first term in the last expression in (A20) is positive and the second term is negative. But we note that\[
E_{p_b} p^{-\lambda} \frac{1}{2} \sigma \frac{1}{2} (\eta_1^- - \eta_2^-)(\eta_1^- - \eta_2^+) + \frac{g\lambda p (\mu_2 - \mu_1)}{\beta - \mu_1}
\]
is a decreasing function of \( p \). From (A19) and the fact that \( u_{p_b} \) is twice differentiable by construction, it follows that \( L^2(u_{p_b}) + f + g(\lambda p - q)^+ \big|_{p=q/\lambda} < 0 \). Therefore,
\[(A21) \quad L^2(u_{p_b}) + f + g(\lambda p - q)^+ < 0 \text{ for } p \geq q/\lambda.\]

Therefore, from (A19) and (A21), \( \dot{L}(u_{p_b}) + f + g(\lambda p - q)^+ < 0 \) for \( p > p_b \).

By the result of Proposition 1 and the fact that the inequality above is strict, it follows that choosing strategy 1 throughout is the unique optimal policy for the manager and that \( u_{p_b} \) is the optimal value function. This completes the proof.

Before proceeding with the proofs of Propositions 4, 5, 6, we prove the following lemma that is used frequently in our proofs.

**Lemma 2**

If \( u \) is defined as in (8) or (9) with \( p_b < r < \infty \) then a necessary and sufficient condition for \( u \) to be twice differentiable at \( p = r \) (and therefore everywhere) is
\[(A22) \quad L^2(u_r) + f + g(\lambda p - q)^+ \big|_{p=r} = 0 \]

Moreover, this is also equivalent to the condition
\[(A23) \quad L(u_r) + f + g(\lambda p - q)^+ \big|_{p=r} = 0 \]

which is in turn equivalent to
\[(A24) \quad L(u_r) + f + g(\lambda p - q)^+ \big|_{p=r} = L^2(u_r) + f + g(\lambda p - q)^+ \big|_{p=r} = 0 \]

**Proof of Lemma 2**
Recall that $u_r$ is the value of the policy of choosing strategy 2 for $p \leq r$ and strategy 1 for $p > r$. Therefore, by construction, $L^2(u_r) + f + g(\lambda p - q)^+ = 0$ for $p < r$.

Hence,

$$L^2(u_r) + f + g(\lambda p - q)^+ \bigg|_{p=r} = 0$$

If (A22) holds, then we easily see by subtracting (A25) from (A22) that

$$\frac{1}{2} \sigma^2 \left[ \frac{d^2}{dp^2} u_r \bigg|_{p=r} - \frac{d^2}{dp^2} u_r \bigg|_{p=r} \right] = 0$$

from which it follows that $u_r$ is twice differentiable at $r$ and therefore everywhere. Conversely, if $u_r$ is twice differentiable at $r$, we easily see from (A25) that (A22) must hold. Conditions (A23) and (A24) can be shown to be equivalent to (A22) by exactly analogous arguments. This completes the proof.

By the result of the above lemma, it therefore suffices to show either (A22) or (A23) in order to show that the value function is twice differentiable.

**Proof of Proposition 4**

We begin by noting that the function $L^2(u_r) + f + g(\lambda p - q)^+ \bigg|_{p=r}$ is a continuous function of $r$. It therefore follows from (11) and (12) that there exists $p_*$ with $p_b < p_* \leq q/\lambda$ such that

$$L^2(u_{p_*}) + f + g(\lambda p - q)^+ \bigg|_{p=p_*} = 0$$

We shall show that $p_*$ is the required optimal switching point. By the result of Proposition 1, we need to show that

$$L^1(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p_b < p < p_*$$

$$L^2(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p > p_*$$

By the result of Lemma 2, (A26) implies that $u_{p_*}$ is twice differentiable at $p = p_*$ and that

$$L^1(u_{p_*}) + f + g(\lambda p - q)^+ \bigg|_{p=p_*} = L^2(u_{p_*}) + f + g(\lambda p - q)^+ \bigg|_{p=p_*} = 0$$

From the definition (8) of $u_{p_*}$, (A26), (A28) imply that
Since $\eta^+ < \eta^- < 0 < \eta^+ < \eta^+$ by hypothesis and $u_p$ must be an increasing function of $p$, the conditions (A29) imply that

$$A_p \cdot \frac{1}{2} \sigma_1^2 (\eta^+_2 - \eta^-_1)(\eta^+_2 - \eta^-_1) \rho. p. \eta^+_1 + B_p \cdot \frac{1}{2} \sigma_1^2 (\eta^-_2 - \eta^+_1)(\eta^-_2 - \eta^+_1) \rho. \eta^-_1 = 0$$

(C29)

$$C_p \cdot \frac{1}{2} \sigma_2^2 (\eta^+_1 - \eta^-_2)(\eta^+_1 - \eta^-_2) \rho. \eta^+_1 + D_p \cdot \frac{1}{2} \sigma_2^2 (\eta^-_1 - \eta^+_2)(\eta^-_1 - \eta^+_2) \rho. \eta^-_1 = 0$$

We now see that $A_p \cdot \frac{1}{2} \sigma_1^2 (\eta^+_2 - \eta^-_1)(\eta^+_2 - \eta^-_1) \rho. \eta^+_1 + B_p \cdot \frac{1}{2} \sigma_1^2 (\eta^-_2 - \eta^+_1)(\eta^-_2 - \eta^+_1) \rho. \eta^-_1$ must be an increasing function of $p$ and

$$C_p \cdot \frac{1}{2} \sigma_2^2 (\eta^+_1 - \eta^-_2)(\eta^+_1 - \eta^-_2) \rho. \eta^+_1 + D_p \cdot \frac{1}{2} \sigma_2^2 (\eta^-_1 - \eta^+_2)(\eta^-_1 - \eta^+_2) \rho. \eta^-_1$$

must be a decreasing function of $p$. Since both the expressions above are equal to zero at $p = p_*$, we easily conclude that

$$L^1(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p_b < p < p_*$$

$$L^2(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p_* < p < q/\lambda$$

We can use arguments identical to those used in the proof of Proposition 3 to show that

$$L^2(u_{p_*}) + f + g(\lambda p - q)^+ =$$

$$E_p \cdot \rho. \eta^+_1 \cdot \frac{1}{2} \sigma_2^2 (\eta^-_1 - \eta^-_2)(\eta^-_1 - \eta^-_2) \rho. \eta^+_1 + \frac{g\lambda p(\mu^-_2 - \mu^-_1)}{\beta - \mu^-_1} < 0 \text{ for } p \geq \frac{q}{\lambda}$$

(A30) and (A31) together imply that (A27) holds. Therefore, $u_{p_*}$ is the optimal value function by the result of Proposition 1 and the policy of switching from strategy 2 to strategy 1 at $p_*$ is optimal.

This completes the proof.

\[ \diamond \]

**Proof of Proposition 5**

We begin by noting that by the definitions (8) and (9) of $u_{q/\lambda}$,

$$L^2(u_{q/\lambda}) + f + g(\lambda p - q)^+ \bigg|_{p = q/\lambda} = 0$$

(A32)

and
Subtracting (A32) from (13), we see that
\[
\frac{1}{2} \sigma^2 \left( \frac{q}{\lambda} \right)^2 \left[ \frac{d^2}{dp^2} (u_{q/I}) \right]_{p=q/I+} - \frac{d^2}{dp^2} (u_{q/I}) \right]_{p=q/I-} > 0
\]
This implies that
\[
\frac{1}{2} \sigma^2 \left( \frac{q}{\lambda} \right)^2 \left[ \frac{d^2}{dp^2} (u_{q/I}) \right]_{p=q/I+} - \frac{d^2}{dp^2} (u_{q/I}) \right]_{p=q/I-} > 0
\]
(A33) and (A34) clearly imply that
\[
L^1(u_{q/I}) + f + g(\lambda p - q)^+ \mid_{p=q/I-} < 0
\]
We need to show the existence of \( p_* > q/\lambda \) such that
\[
L^1(u_{p_*}) + f + g(\lambda p - q)^+ \mid_{p=p_*} = 0
\]
which would imply by the result of Lemma 2, that \( u_{p_*} \) is twice differentiable at \( p_* \) and that
\[
L^1(u_{p_*}) + f + g(\lambda p - q)^+ \mid_{p=p_*} = 0
\]
We prove this by first showing that
\[
\lim_{r \to \infty} L^1(u_r) + f + g(\lambda p - q)^+ \mid_{p=r} = \infty
\]
As \( r \to \infty \), the value function \( u_r \) clearly approaches the value function \( u_* \) of the policy of choosing strategy 2 throughout. It is easy to see that the functional form of \( u_* \) is
\[
u_* (p) = A_* p^{\eta_*} + B_* p^{\eta_*} + \frac{f}{\beta} ; p < \frac{q}{\lambda} \\
u_* (p) = C_* p^{\eta_*} + \frac{f - gq}{\beta} + \frac{g\lambda p}{\beta - \mu_2} ; p > \frac{q}{\lambda} \\
u_* (p_b) = 0
\]
We now note that
\[
\lim_{p \to \infty} L^1(u_*) + f + g(\lambda p - q)^+ = \\
\lim_{p \to \infty} \left[ C_* p^{\eta_*} \left( \frac{1}{2} \sigma^2 (\eta_2^*)^2 + (\mu_2 - \frac{1}{2} \sigma^2 \eta_2^* - \beta) \right) + \frac{g\lambda p (\mu_2 - \mu_2)}{\beta - \mu_2} \right] = \infty
\]
as \( \eta_2^2 < 0 \). The result (A40) implies that (A38) holds. It now easily follows by continuity that (A36) holds and therefore (A37) holds by the result of Lemma 2. We shall now show that \( p_* \) is the required "optimal switching point" where \( u_{p_*} \) is defined by (9). By the result of Proposition 1, we clearly need to show that

\[
L'(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p_b < p < p_*
\]

(A41)

\[
L^2(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p > p_*
\]

For \( p > p_* \),

(A42) \[
L^2(u_{p_*}) + f + g(\lambda p - q)^+ = E_{p_*} \frac{1}{2} \sigma_2^2 p^{n_{1^-}} (\eta_{1^-} - \eta_{2^-})(\eta_{1^-} - \eta_{2^+}) + \frac{g\lambda p(\mu_2 - \mu_1)}{\beta - \mu_1}
\]

Since \( \mu_1 > \mu_2 \) and \( \eta_{2^-} < \eta_{1^-} < 0 < \eta_{1^+} < \eta_{2^+} \) by hypothesis, (A37) holds only if

(A43) \[
E_{p_*} < 0
\]

It now follows that the expression \( E_{p_*} \frac{1}{2} \sigma_2^2 p^{n_{1^-}} (\eta_{1^-} - \eta_{2^-})(\eta_{1^-} - \eta_{2^+}) + \frac{g\lambda p(\mu_2 - \mu_1)}{\beta - \mu_1} \) is a decreasing function of \( p \). Therefore, (A37) implies that

(A44) \[
L^2(u_{p_*}) + f + g(\lambda p - q)^+ < 0 \text{ for } p > p_*
\]

Using the fact that \( u_{p_*} \) is twice differentiable at \( p = p_* \), we can show that (A43) implies that the coefficients \( C_{p_*}, D_{p_*} \) in the definition (9) of \( u_{p_*} \) satisfy

(A45) \[
C_{p_*} > 0, D_{p_*} < 0
\]

Now, using the fact that \( u_{p_*} \) is twice differentiable at \( p = q/\lambda \), we can show that the coefficients \( A_{p_*}, B_{p_*} \) in the definition (9) of \( u_{p_*} \) satisfy

(A45) \[
A_{p_*} > 0, B_{p_*} < 0
\]

We can use arguments identical to those we have been using so far to show that

\[ L'(u_{p_*}) + f + g(\lambda p - q)^+ \text{ is an increasing function of } p \text{ for } p < p_* \]. (A36) and (A37) now clearly imply that

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18 Strictly this needs to be shown rigorously, but the arguments are quite straightforward and are available from the author upon request.
We have therefore shown that (A41) holds and hence, the hypotheses of Proposition 1 are satisfied. Therefore, $u_{p_\ast}$ is the optimal value function and the policy of switching from strategy 2 to strategy 1 at $p_\ast$ is optimal. This completes the proof.

\[ \text{Proof of Proposition 6} \]

We shall use arguments similar to those used in the proofs of Propositions 3, 4 and 5 in the previous section. We consider the set of policies defined by the parameter $r$ with $p_b \leq r \leq \frac{\vartheta}{\lambda}$ where the firm chooses the high volatility strategy for $p \leq r$ and the low volatility strategy for $p \geq r$. We denote the corresponding value function by $u_r$. We shall then show that the required “optimal switching level” $p^\ast$ is given by

\[ p^\ast = r^\ast \text{ if } L^1(u_{r^\ast}) + \lambda p + \tau \lambda p \big|_{p=r^\ast} = L^2(u_{r^\ast}) + \lambda p + \tau \lambda p \big|_{p=r} = 0 \text{ and } r^\ast > p_b \]

\[ p^\ast = p_b \text{ if such an } r^\ast \text{ does not exist} \]

Therefore, the optimal policy for the firm is to choose the low volatility strategy throughout if the first condition in (A47) does not hold.

**Step 1**: Derivation of $u_r$ for fixed $r$ such that $p_b \leq r \leq \frac{\vartheta}{\lambda}$.

As in (8), we can express the functional form of the value function $u_r$ as follows:

\[ u_r(p) = A_r p^{\rho^r_+} + B_r p^{\rho^r_-} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta}; p_b < p \leq r \]

\[ = C_r p^{\rho^r_+} + D_r p^{\rho^r_-} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta}; r < p \leq \frac{\vartheta}{\lambda} \]

\[ = E_r p^{\rho^r_+} + \frac{\lambda p}{\delta} + \frac{\tau \vartheta}{r}; p > \frac{\vartheta}{\lambda} \]

\[ u_r(p_b) = (1-\alpha) \frac{\lambda p_b}{\delta} \]

As $p \to \infty$, clearly $u(p) \to \frac{\lambda p}{\delta} + \frac{\tau \vartheta}{r}$ since the firm is profitable with probability very close to one.

Further, $\frac{\lambda p}{\delta} + \frac{\tau \vartheta}{r}$ is the value of cash flows associated with the firm in the hypothetical situation when it enjoys tax advantages of debt whether it is profitable or not and faces no bankruptcy costs.
In reality, it faces the possibility of bankruptcy and its tax shield is only \( \tau \lambda p \leq \tau \delta \) for \( p \leq \delta / \lambda \).

Therefore, in (A48), we must have \( E_r < 0 \). In other words, the value function of any policy is always strictly less than \( \frac{\lambda p}{\delta} + \frac{\tau \delta}{r} \). Therefore, for \( p > \frac{\delta}{\lambda} \),

\[
L^1(u_r) + \lambda p + \tau q = \frac{1}{2} \sigma^2 E_r (\rho^- - \rho^+) (\rho^- - \rho^+) p^{\rho^-} < 0
\]

In particular,

\[
L^1(u_r) + \lambda p + \tau \delta \bigg|_{p=\delta / \lambda} < 0 \text{ for all values of } r \text{ such that } p_b \leq r \leq \delta / \lambda.
\]

By the smooth matching of the value function at \( p = \delta / \lambda \), it therefore follows that

\[
L^1(u_r) + \lambda p + \tau \delta p \bigg|_{p=\delta / \lambda} < 0
\]

We now have two possible scenarios.

**Case 1:**

Suppose

\[
L^1(u_{p_b}) + \lambda p + \tau \delta \bigg|_{p=p_b} \leq 0
\]

We shall show that choosing strategy 2 throughout is optimal for the firm. By the result of **Proposition 1**, we only need to show that

\[
L^1(u_{p_b}) + C(p) \leq 0 \text{ for all } p \geq p_b.
\]

For \( p \geq \delta / \lambda \), the above follows from (A49). For \( p_b \leq p \leq \delta / \lambda \), we have from (A48)

\[
L^1(u_{p_b}) + C(p) = L^1(C_{p_b} p^{\rho^+} + D_{p_b} p^{\rho^-} + \frac{\lambda p}{\delta} + \frac{\tau \delta p}{\delta}) + \delta p + \tau \delta p
\]

\[
= \frac{1}{2} \sigma^2 \left[ C_{p_b} (\rho^+ - \rho^-) (\rho^+ - \rho^-) p^{\rho^+} + D_{p_b} (\rho^- - \rho^+) (\rho^- - \rho^-) p^{\rho^-} \right]
\]

By the smooth matching of value functions at \( p = \delta / \lambda \) and some algebra, we can show that

\[
C_{p_b} = \frac{\tau \delta}{(\rho^+ - \rho^-)} \left[ (r - \delta) \rho^- - r \right] \frac{\delta}{r \delta} \frac{\rho^-}{\rho^-} < 0
\]

since \( r > \delta, \rho^- < 0 \). If \( D_{p_b} \leq 0 \), then from (A54) we easily see that \( L^1(u_{p_b}) + C(p) < 0 \)

---

19 **Note:** \( u_{p_b} \) is the value function of the policy that chooses strategy 2 for all values above bankruptcy.
as required since $\rho^-_2 < \rho^-_1 < \rho^+_1 < \rho^+_2$ from (16). Therefore (A53) is true and the policy of choosing strategy 2 throughout is optimal.

On the other hand, if $D_{p_b} > 0$, then we see that the last term in (A54) is a decreasing function of $p$. Therefore, (A52) clearly implies that (A53) is true and the policy of choosing strategy 2 throughout is optimal. Therefore, it is optimal for the firm to choose strategy 2 throughout if (A52) holds.

**Case 2:**

Suppose \( L^1(u_{p_b}) + \lambda p + \tau \lambda p \bigg|_{p=p_b} > 0 \).

From (A50) and an application of the intermediate value theorem and continuity, we see that there exists a value \( r^* \) with $p_b \leq r^* < \theta / \lambda$ such that

\[
L^1(u_{r^*}) + \lambda p + \tau \lambda p \bigg|_{p=r^*} = 0. \tag{A56}
\]

We shall show that \( r^* \) is the required “optimal switching level” \( p^* \). By the result of Lemma 2, (A56) implies that \( u_{r^*} \) is twice continuously differentiable at \( p = r^* \) and that

\[
L^1(u_{r^*}) + \lambda p + \tau \lambda p \bigg|_{p=r^*} = L^2(u_{r^*}) + \lambda p + \tau \lambda p \bigg|_{p=r^*} = 0. \tag{A57}
\]

In order to show the optimality of this policy, and since (A49) holds, it remains to show by the result of Proposition 1 that

\[
L^1(u_{r^*}) + \lambda p + \tau \lambda p < 0 \text{ for } r^* < p \leq \theta / \lambda
\]
\[
L^2(u_{r^*}) + \lambda p + \tau \lambda p < 0 \text{ for } p_b \leq p < r^*
\]

We see that for \( r^* \leq p \leq \theta / \lambda \),

\[
L^1(u_{r^*}) + C(p) = L^1(C^\rho_{r^*}) + D_{r^*} p^\rho_{r^*} + \frac{\lambda p}{\delta} + \frac{\tau \lambda p}{\delta} + \lambda p + \tau \lambda p
\]
\[
= \frac{1}{2} \sigma^2_1 (C^\rho_{r^*}, p^\rho_{r^*}(\rho^+_2 - \rho^-_1)(\rho^+_1 - \rho^-_2) + D_{r^*} p^\rho_{r^*}(\rho^-_2 - \rho^+_1)(\rho^-_1 - \rho^+_2)
\]

By the smooth matching of the value functions at \( p = \theta / \lambda \), we have

\[\text{Note: In the above, } u_{r^*} \text{ is the value of the policy of choosing strategy 1 for } p \leq r^* \text{ and strategy 2 for } p > r^*.\]
From (A60) and the fact that \( \rho_2^- < \rho_1^- < \rho_1^+ < \rho_2^+ \) (16), we see that the first term in the last expression in (A59) is negative for all \( p \). Therefore, (A57) can only hold if

\[
(A61) \quad D_r^* > 0.
\]

In this case, we see from (A59) and the fact that \( \rho_2^- < \rho_1^- < \rho_1^+ < \rho_2^+ \) (16) that \( L^1(u_r^*) + C(p) \) is a decreasing function of \( p \) for \( r^* \leq p \leq \bar{\theta} / \bar{\lambda} \). Therefore, (A57) implies that

\[
L^1(u_r^*) + C(p) = L^1(u_r^*) + \lambda p + \tau \lambda p < 0 \quad \text{for} \quad r^* < p < \bar{\theta} / \bar{\lambda}.
\]

It only remains to show that \( L^2(u_r^*) + \lambda p + \tau \lambda p < 0 \) for \( p_b \leq p < r^* \). By (A58), we have

\[
L^2(u_r^*) + \lambda p + \tau \lambda p = \left. L^2(u_r^*) + \lambda p + \tau \lambda p \right|_{r^*} = 0. \quad \text{By (A48), for} \quad p_b \leq p < r^*, \quad
\]

\[
(A62) \quad L^2(u_r^*) + \lambda p + \tau \lambda p = \frac{1}{2} \sigma_2^2 [A_r^* (\rho_1^+ - \rho_2^+)(\rho_1^+ - \rho_2^-) p^{\rho_1^+} + B_r^* (\rho_1^- - \rho_2^+)(\rho_1^- - \rho_2^-) p^{\rho_1^-}]
\]

From the smooth matching of value functions at \( p = r^* \) and some tedious algebra, we obtain

\[
(A63) \quad A_r^* = \left[ \frac{\rho_2^+ - \rho_1^-}{\rho_1^+ - \rho_1^-} \right] C_r^* (r^*)^{\rho_1^- - \rho_1^+} + \left[ \frac{\rho_2^- - \rho_1^-}{\rho_1^+ - \rho_1^-} \right] D_r^* (r^*)^{\rho_1^- - \rho_1^+}
\]

Since \( \rho_2^- < \rho_1^- < \rho_1^+ < \rho_2^+ \) from (16) and \( C_r^* < 0, D_r^* > 0 \) from (A60) and (A61), we easily see that \( A_r^* < 0 \) from the above expression.

From (A62) and the fact that \( \rho_2^- < \rho_1^- < \rho_1^+ < \rho_2^+ \), we now see that since

\[
L^2(u_r^*) + \lambda p + \tau \lambda p < 0 \quad \text{at} \quad p = r^*, \quad \text{we must have} \quad B_r^* > 0. \quad \text{We now see from (A62) that}
\]

\[
L^2(u_r^*) + \lambda p + \tau \lambda p \quad \text{is a decreasing function of} \quad p \quad \text{for} \quad p_b \leq p \leq r^*. \quad \text{Therefore, since}
\]

\[
L^2(u_r^*) + \lambda p + \tau \lambda p < 0 \quad \text{at} \quad p = r^*, \quad \text{we see that} \quad L^2(u_r^*) + \lambda p + \tau \lambda p < 0 \quad \text{for} \quad p_b \leq p < r^*.
\]

Therefore, by the result of Proposition 2, the policy of choosing strategy 1 for \( p \leq r^* \) and strategy 2 for \( p > r^* \) is optimal. This completes the proof. ♦
References


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**TABLE 2: AGENCY COSTS AND OPTIMAL LEVERAGE**

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